L^2-invariants and 3-manifolds

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Bonn, 14. & 19. June

Outline

- We introduce the universal L^2-torsion.
- We introduce twisted L^2-torsion and relate it to the Thurston norm.
- We relate the universal L^2-torsion to the dual of the Thurston polytop.
- We relate the L^2-Euler characteristic to the Thurston norm and discuss a question about knot theory by Simon.

- It will not be necessary to recall the definitions of L^2-Betti numbers and L^2-torsion for this lecture. It suffices to recall the basic results from previous lectures.

Definition (Admissible 3-manifold)

A 3-manifold M is called admissible if it is connected, compact, orientable, and prime, has infinite fundamental group, its boundary is empty or a union of tori, and is not homeomorphic to $S^1 \times D^2$ or $S^1 \times S^2$.

- One example is the manifold M_K with boundary a 2-torus which is obtained by deleting the interior of a tubular neighborhood of a non-trivial knot $K \subseteq S^3$.

Theorem (Lott-Lück [12])

For every admissible 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ of its universal covering with respect to the $\pi_1(M)$-action vanish.

- We are interested in the case where all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion $\rho^{(2)}(\widetilde{M})$, which is a real number.

Theorem (Lück-Schick [15])

Let M be an admissible 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

Universal L^2-torsion

- There is a zoo of powerful invariants of admissible 3-manifolds such as L^2-invariants and their twisted versions, Alexander polynomials and their twisted versions, Thurston polytopes, and also of invariants of high-dimensional manifolds. They all take different values in different groups, but do share some common properties such as sum formulas, fibration formulas, Poincaré duality and so on.

- This gives the impression that a universal invariant is lurking in the background which encompasses all of them.

- More precisely, we want to introduce a very general invariant which also has these properties and from which all the other invariants are obtained by specifying appropriate homomorphisms between the groups, where they take values in.

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced bounded G-equivariant $L^2(G)^m \to L^2(G)^m$operator is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.
- relations

$$\begin{aligned} \left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2]; \\ [g \circ f] &= [f] + [g]. \end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

- An injective group homomorphism $i\colon H \to G$ induces a homomorphism

$$i_*\colon \operatorname{Wh}^w(H) \to \operatorname{Wh}^w(G),$$

since $i_* f$ is weak isomorphism if f is a weak isomorphism.

- The condition that i is injective is necessary.

- There is an obvious group homomorphism

$$\phi\colon \operatorname{Wh}(G) \to \operatorname{Wh}^w(G)$$

coming from the fact that for a $\mathbb{Z}G$-automorphism $f\colon \mathbb{Z}G^n \to \mathbb{Z}G^n$ the induced operator $f^{(2)}$ is obviously a weak isomorphism.

Definition (Weak chain contraction)

Consider a $\mathbb{Z}G$-chain complex C_*. A weak chain contraction (γ_*, u_*) for C_* consists of a $\mathbb{Z}G$-chain map $u_* \colon C_* \to C_*$ and a $\mathbb{Z}G$-chain homotopy $\gamma_* \colon u_* \simeq 0_*$ such that $u_*^{(2)} \colon C_*^{(2)} \to C_*^{(2)}$ is a weak isomorphism for all $n \in \mathbb{Z}$ and $\gamma_n \circ u_n = u_{n+1} \circ \gamma_n$ holds for all $n \in \mathbb{Z}$.

Definition (Universal L^2-torsion)

Let C_* be a finite based free $\mathbb{Z}G$-chain complex such that C_* is weakly acyclic, i.e., all L^2-Betti numbers of the finite Hilbert $\mathcal{N}(G)$-chain complex $C_*^{(2)} = L^2(G) \otimes_{\mathbb{Z}G} C_*$ vanish. Define its universal L^2-torsion

$$\rho_u^{(2)}(C_*) \in \widetilde{K}_1^w(\mathbb{Z}G)$$

by

$$\rho_u^{(2)}(C_*) = [(uc+\gamma)_{\mathrm{odd}}] - [u_{\mathrm{odd}}],$$

where (γ_*, u_*) is any weak chain contraction of C_*.
We sometimes pass to $\mathrm{Wh}^w(G)$ without further notice.

- Here is a motivation for the weak chain contraction and the definition above.
- Let R be a ring and C_* be an acyclic finite based free R-chain complex. If γ_* is a chain contraction, we had defined its torsion $\tau(C_*) \in \widetilde{K}_1(R)$ by the class of the isomorphism of finitely generated based free R-modules $(c+\gamma)_{\mathrm{odd}} \colon C_{\mathrm{odd}} \to C_{\mathrm{ev}}$.
- Let $u_* \colon C_* \xrightarrow{\cong} C_*$ be an R-chain automorphism of C_*. Then (δ_*, u_*) for $\delta_n = \gamma_n \circ u_n$ is not a chain contraction anymore, but satisfies $c_{n+1} \circ \delta_n + \delta_n \circ c_n = u_n$ for $n \in \mathbb{Z}$, so is something like a weak chain contraction.
- Now one computes in $\widetilde{K}_1(R)$

$$\begin{aligned}
[(uc+\delta)_{\mathrm{odd}}] - [u_{\mathrm{odd}}] &= [(cu+\gamma\circ u)_{\mathrm{odd}}] - [u_{\mathrm{odd}}] \\
&= [(c+\gamma)_{\mathrm{odd}} \circ u_{\mathrm{odd}}] - [u_{\mathrm{odd}}] = [(c+\gamma)_{\mathrm{odd}}] + [u_{\mathrm{odd}}] - [u_{\mathrm{odd}}] \\
&= [(c+\gamma)_{\mathrm{odd}}] = \tau(C_*).
\end{aligned}$$

Example (The combinatorial Laplace operator)

- Let C_* be a weakly acyclic finite based free $\mathbb{Z}G$-chain complex.

- Define its *n-th combinatorial Laplace operator* to be the $\mathbb{Z}G$-chain map
$$\Delta_n^c := c_{n+1} \circ c_n^* + c_n^* \circ c_n \colon C_n \to C_n,$$
where $c_n^* \colon C_{n-1} \to C_n$ is the adjoint of c_n with respect to the standard involution on $\mathbb{Z}G$.

- Then $\mathrm{id}_{L^2(G)} \otimes_{\mathbb{Z}G} \Delta_n \colon L^2(G) \otimes_{\mathbb{Z}G} C_* \to L^2(G) \otimes_{\mathbb{Z}G} C_*$ is the n-th Laplace operator.

- Thus we obtain an explicite weak chain contraction (γ_*, u_*) of C_* if we put $\gamma_n = c_n^*$ and $u_n = \Delta_n^c$.

Example (Trivial G)

- Suppose that G is trivial.

- Then $K_1^w(\mathbb{Z}) \to K_1(\mathbb{Q})$ sending the class of $f\colon \mathbb{Z}^n \to \mathbb{Z}^n$ to the class of $\mathrm{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} f$ is an isomorphism, since for any automorphism $g\colon \mathbb{Q}^n \to \mathbb{Q}^n$ there exists a natural number $k \in \mathbb{N}$ and a $\mathbb{Z}$-map $f\colon \mathbb{Z}^n \to \mathbb{Z}^n$ with $k \cdot g = \mathrm{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} f$. We obtain an isomorphism

$$\alpha\colon \ \mathrm{Wh}^w(\{1\}) \xrightarrow{\cong} \widetilde{K}_1(\mathbb{Q}) \xrightarrow{|\det|} \mathbb{Q}^{>0}.$$

- The following assertions are equivalent for a finite free $\mathbb{Z}$-chain complex C_*:
  - C_* is weakly acyclic;
  - $\mathbb{R} \otimes_{\mathbb{Z}} C_*$ is acyclic;
  - $\mathbb{Q} \otimes_{\mathbb{Z}} C_*$ is acyclic;
  - $H_n(C_*)$ is finite for every $n \in \mathbb{Z}$.

Example (continued)

- Let $\gamma_*\colon \mathbb{Q} \otimes_{\mathbb{Z}} C_* \to \mathbb{Q} \otimes_{\mathbb{Z}} C_*$ be a $\mathbb{Q}$-chain contraction. Choose a natural number k such that for every $n \in \mathbb{Z}$ there is a $\mathbb{Z}$-map $\delta_n\colon C_n \to C_{n+1}$ with $k \cdot \gamma = \mathrm{id}_{\mathbb{Q}} \otimes_{\mathbb{Z}} \delta_n$. Then $(\delta_*, k \cdot \mathrm{id}_{C_*})$ is a weak $\mathbb{Z}$-chain contraction.

- Now one easily checks for a weakly acyclic finite based free $\mathbb{Z}$-chain complex that $\rho_u^{(2)}(C_*)$ is the Milnor torsion, i.e.,

$$\alpha\bigl(\rho_u^{(2)}(C_*)\bigr) = \prod_{n \in \mathbb{Z}} |H_n(C_*)|^{(-1)^n}.$$

- Fix a group G. An additive L^2-torsion invariant (A, a) consists of an abelian group A and an assignment which associates to a weakly acyclic finite based free $\mathbb{Z}G$-chain complex C_* an element

$$a(C_*) \in A$$

such that for any based exact short sequence of such $\mathbb{Z}G$-chain complexes $0 \to C_* \to D_* \to E_* \to 0$ we get

$$a(D_*) = a(C_*) + a(E_*),$$

and we have for each trivial unit $\pm g \in \mathbb{Z}G$

$$a\bigl(\cdots \to 0 \to \mathbb{Z}G \xrightarrow{\pm g} \mathbb{Z}G \to 0 \to \cdots\bigr) = 0.$$

- We call an additive L^2-torsion invariant (U, u) universal if for every additive L^2-torsion invariant (A, a) there is precisely one group homomorphism $f\colon U \to A$ satisfying $f(u(C_*)) = a(C_*)$ for any such $\mathbb{Z}G$-chain complex.

Lemma

$(\mathrm{Wh}^w(\mathbb{Z}G), \rho_u^{(2)})$ *is the universal additive* L^2*-torsion invariant for* G*.*

- The proof of this lemma is not hard, it is essentially a consequence of the additivity relation appearing in the definition of $K_1^w(\mathbb{Z}G)$.
- Of course universal is to be understood that our invariant for manifolds can be read off from the cellular chain complex of an appropriate covering. This is true for many but certainly not for all invariants of 3-manifolds. We will give many examples below.

Lemma

The universal L^2-torsion is a simple homotopy invariant. More precisely, for the group homomorphism introduced above

$$\phi\colon \operatorname{Wh}(G) \to \operatorname{Wh}^w(G)$$

and for every $\mathbb{Z}G$-chain homotopy equivalence $f_*\colon C_* \to D_*$ of weakly acyclic finite based free $\mathbb{Z}G$-chain complexes, we get in $\operatorname{Wh}^w(G)$

$$\rho_u^{(2)}(D_*) - \rho_u^{(2)}(C_*) = \phi(\tau(f_*)).$$

Proof.

- Let $f_*\colon C_* \to D_*$ be a $\mathbb{Z}G$-chain homotopy equivalence. Then $\operatorname{cone}(f_*)$ is an acyclic finite based free $\mathbb{Z}G$-chain complex. Let γ_* be a $\mathbb{Z}G$-chain contraction. It can be viewed as a weak chain contraction $(\gamma_*, \operatorname{id}_*)$. Unraveling the definitions shows

$$\phi(\tau(f_*)) = \rho_u^{(2)}(\operatorname{cone}(f_*)) \in \operatorname{Wh}^w(G).$$

- We deduce from the short exact sequence of weakly acyclic finite based free $\mathbb{Z}G$-chain complexes

$$0 \to D_* \to \operatorname{cone}(f_*) \to \Sigma C_* \to 0$$

that we get in $\operatorname{Wh}^w(G)$

$$\begin{aligned}\phi(\tau(f_*)) = \rho_u^{(2)}(\operatorname{cone}(f_*)) = \rho_u^{(2)}(D_*) + \rho_u^{(2)}(\Sigma C_*)\\ = \rho_u^{(2)}(D_*) - \rho_u^{(2)}(C_*).\end{aligned}$$

Definition (Universal L^2-torsion for spaces)

Let X be a connected finite CW-complex with fundamental group π. Suppose that $\widetilde{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\widetilde{X}) = 0$ for all $n \in \mathbb{Z}$. Define the universal L^2-torsion

$$\rho_u^{(2)}(\widetilde{X}) \in \mathrm{Wh}^w(\pi_1(X))$$

to be the universal L^2-torsion of the weakly acyclic finite based free $\mathbb{Z}\pi$-chain complex $C_*(\widetilde{X})$.

This definition extends in the obvious way to a (not necessarily connected) finite CW-complex by

$$\begin{aligned} \mathrm{Wh}^w(\pi(X)) &:= \bigoplus_{C \in \pi_0(X)} \mathrm{Wh}^w(\pi(C)), \\ \rho_u^{(2)}(\widetilde{X}) &:= \{\rho_u^{(2)}(\widetilde{C}) \mid C \in \pi_0(C)\}. \end{aligned}$$

- The next result is a direct consequence of the lemma above.

Theorem (Simple homotopy invariance)

Consider a homotopy equivalence $f\colon X \to Y$ of finite CW-complexes. Suppose that $\widetilde{X}$ or $\widetilde{Y}$ is L^2-acyclic.

Then $\widetilde{X}$ and $\widetilde{Y}$ are L^2-acyclic and we get in $\mathrm{Wh}^w(\pi)$

$$\rho_u^{(2)}(\widetilde{Y}) - \rho_u^{(2)}(\widetilde{X}) = \phi(\tau(f)).$$

Theorem (Sum formula)

Let X be a finite CW-complex with subcomplexes X_0, X_1 and X_2 satisfying $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Suppose $\widetilde{X_0}$, $\widetilde{X_1}$ and $\widetilde{X_2}$ are L^2-acyclic and the inclusions $X_i \to X$ are π-injective.

Then $\widetilde{X}$ is L^2-acyclic and we get

$$\rho_u^{(2)}(\widetilde{X}) = (i_1)_*\rho_u^{(2)}(\widetilde{X_1}) + (i_2)_*\rho_u^{(2)}(\widetilde{X_2}) - (i_0)_*\rho_u^{(2)}(\widetilde{X_0}),$$

where $(i_k)_* \colon \mathrm{Wh}^w(\pi(X_k)) \to \mathrm{Wh}(\pi(X))$ is induced by the inclusion $i_k \colon X_k \to X$.

Theorem (Fibration formula)

Let $F \to E \to B$ be a fibration of connected finite CW-complexes such that $\widetilde{F}$ is L^2-acyclic and the inclusion $F \to E$ is π-injective.

Then $\widetilde{E}$ is L^2-acyclic and we get

$$\rho_u^{(2)}(\widetilde{E}) = \chi(B) \cdot i_* \rho_u^{(2)}(\widetilde{F}),$$

where $i_* \colon \mathrm{Wh}(\pi(F)) \to \mathrm{Wh}(\pi(E))$ is induced by the inclusion $i \colon F \to X$.

Example

- If $S^1 \to E \to B$ is a fibration of connected CW-complexes such that $\pi_2(B)$ and $\chi(B)$ vanishes, then by the result above $\widetilde{E}$ is L^2-acyclic and $\rho_u^{(2)}(\widetilde{E})$ vanishes.
- In particular $\widetilde{T^n}$ vanishes for $n \geq 2$.

Theorem (Poincaré duality)

Let M be a closed orientable n-dimensional manifold such that $\widetilde{M}$ is L^2-acyclic.

Then

$$\rho_u^{(2)}(\widetilde{M}) = (-1)^{n+1} \cdot *\big(\rho_u^{(2)}(\widetilde{M})\big),$$

where the involution $* \colon \operatorname{Wh}^w(\pi) \to \operatorname{Wh}^w(\pi)$ comes from the standard involution of rings on $\mathbb{Z}\pi$ sending $\sum_{w \in \pi} n_w \cdot w$ to $\sum_{w \in \pi} n_w \cdot w^{-1}$.

Proof.

This comes from the Poincaré $\mathbb{Z}G$-chain homotopy equivalence

$$- \cap [M] \colon C^{n-*}(\widetilde{M}) \xrightarrow{\simeq} C_*(\widetilde{M}).$$

□

Theorem (Realizability of the universal L^2-torsion for closed manifolds, Friedl-Lück [5])

Let G be a group such that there exists a connected CW-complex X with $G = \pi_1(X)$ such that $\widetilde{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\widetilde{X}) = 0$ for every $n \in \mathbb{Z}$. Consider any element $\omega \in \mathrm{Wh}^w(G)$ and any integer $d \geq 2 \cdot \max\{\dim(X), 4\} + 1$.

Then there exists a connected closed smooth d-dimensional manifold M such that $\widetilde{M}$ is L^2-acyclic and we have

$$\rho_u^{(2)}(M; \mathcal{N}(G)) = (-1)^{d+1} \cdot *(\omega) + \omega.$$

- If π is finite, then a variant of $\rho_u^{(2)}(\widetilde{X})$ is closely related to the classical Reidemeister torsion.
- We have $\rho_u^{(2)}(\widetilde{S^1}) = (z-1)$ in $\mathrm{Wh}^w(\mathbb{Z}) \cong \mathbb{Q}(z^{\pm 1})^\times/\{\pm z^n \mid n \in \mathbb{Z}\}$.
- We have already proved $\rho_u^{(2)}(\widetilde{T^n}) = 0$ for $n \geq 2$.

Theorem (Jaco-Shalen-Johannson decomposition)

Let M be an admissible 3-manifold. Let $M_1, M_2, \ldots, M_r$ be its pieces in the Jaco-Shalen-Johannson decomposition. Let $j_i \colon \pi_1(M_i) \to \pi_1(M)$ be the injection induced by the inclusion $M_i \to M$.

Then each M_i and M are L^2-acyclic and we have

$$\rho_u^{(2)}(\widetilde{M}) = \sum_{i=1}^{r} (j_i)_*\bigl(\rho_u^{(2)}(\widetilde{M_i})\bigr).$$

- One can compute the universal L^2-torsion for the Seifert pieces in terms of the values for $\rho_u^{(2)}(S^1)$ using the Seifert fiber structure. The hyperbolic pieces are much harder.

- Many other invariants come from the universal L^2-torsion by applying a homomorphism $K_1^w(\mathbb{Z}G) \to A$ of abelian groups.
- For instance, the Fuglede-Kadison determinant defines a homomorphism
$$\det^{(2)} \colon \mathrm{Wh}^w(\mathbb{Z}G) \to \mathbb{R}$$
which maps the universal L^2-torsion $\rho_u^{(2)}(\widetilde{X})$ to the L^2-torsion $\rho^{(2)}(\widetilde{X})$.
- This already shows that $\rho_u^{(2)}$ is an interesting invariant. Moreover, the formulas for $\rho_u^{(2)}$ imply the corresponding formulas for $\rho^{(2)}$.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[d] & \mathcal{N}(G) \arrow[d] \\ \mathcal{D}(G) \arrow[r] & \mathcal{U}(G) \end{tikzcd}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{R}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is invertible over $\mathcal{U}(G)$, is already invertible over $\mathcal{D}(G)$.

- If G is finite, it is given by

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[d] & \mathbb{C}G \arrow[d, "\mathrm{id}"] \\ \mathbb{Q}G \arrow[r] & \mathbb{C}G \end{tikzcd}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}] \arrow[r] \arrow[d] & L^{\infty}(S^1) \arrow[d] \\ \mathbb{Q}[\mathbb{Z}]_{(0)} \arrow[r] & L(S^1) \end{tikzcd}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.
- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.
- The Atiyah Conjecture implies for a torsionfree group G that the rational group ring has no non-trivial zero-divisors.

Theorem (Linnell [10], Schick [16])

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.*
   Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture, actually even over $\mathbb{C}$.
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This theorem and the proof of the Farrell-Jones Conjecture for the fundamental group of any 3-manifold by Bartels-Farrell-Lück [1] show for the fundamental group π of an admissible 3-manifold (with the exception of some graph manifolds) that it both satisfies the Atiyah Conjecture and that $\mathrm{Wh}(\pi)$ vanishes.

- Let $\mathcal{R}(G)$ be the rational closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every $n \in \mathbb{N}$ and element in $A \in M_{n,n}(\mathcal{D}(G))$, which is invertible over $\mathcal{U}(G)$, is already invertible over $\mathcal{D}(G)$ itself.

- The Atiyah Conjecture predicts $\mathcal{D}(G) = \mathcal{R}(G)$.

Lemma

The following assertions are equivalent for a $\mathbb{Z}G$-homomorphism $f \colon \mathbb{Z}G^n \to \mathbb{Z}G^n$.

- *f is L^2-acyclic, i.e., the induced operator $f^{(2)} \colon L^2(G) \to L^2(G)^n$ is a weak isomorphism;*

- *The induced map $\operatorname{id}_{\mathcal{U}(G)} \colon \mathcal{U}(G)^n \to \mathcal{U}(G)^n$ is bijective;*

- *The induced map $\operatorname{id}_{\mathcal{D}(G)} \colon \mathcal{D}(G)^n \to \mathcal{D}(G)^n$ is bijective.*

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück [14])

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.

- Given any skew-field K, there is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(K) \xrightarrow{\cong} (K^\times)_{\text{abel}} := K^\times/[K^\times, K^\times].$$

- It has the expected properties of a determinant such as additivity and compatibility with multiplication.
- It is defined for an invertible $(1,1)$ -matrix (a) by sending a to its class in $K^\times/[K^\times, K^\times]$.
- If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a $(2,2)$-matrix over a skew-field K, its Dieudonné determinant in $K^\times/[K^\times, K^\times]$ is defined to be the class of $-cb$ if $a = 0$ and to be the class of $ad - aca^{-1}b$ otherwise.
- It is defined inductively over the size of the matrix under consideration by bringing it into upper triangular form by elementary row and column operations and the applying additivity.

Example

- It can happen that for an (r,r)-matrix A over $\mathbb{Z}G$ which is invertible over $\mathcal{D}(G)$ the standard representative of the Dieudonné determinant does not belong to $\mathbb{Z}G$ as the following example due to Linnell shows.

- Let G be the metabelian group

$$\mathbb{Z} \wr \mathbb{Z} = \langle x_i, y \mid x_i x_j = x_j x_i, y^{-1} x_i y = x_{i+1} \text{ for all } i, j \in \mathbb{Z} \rangle.$$

- We have $\mathbb{Q}G \subset L^1(G) \subset \mathcal{U}(G)$, and $\mathcal{D}(G) \subseteq \mathcal{U}(G)$.

- Consider the element $2 - x_0 \in \mathbb{Z}G$. Then we can consider the element $(2 - x_0)y(2 - x_0)^{-1}$ in $\mathcal{D}(G)$. It agrees with $y(1 - x_1/2)(1 - x_0/2)^{-1}$.

Example (Continued)

- We get in the Banach algebra $L^1(G)$ the equality $(1 - x_0/2) \cdot (1 + x_0/2 + x_0^2/4 + \cdots) = 1$. Hence the element $(2 - x_0)y(2 - x_0)^{-1}$ in $\mathcal{U}(G)$ agrees with the element $y(1 - x_1/2)(1 + x_0/2 + x_0^2/4 + \cdots)$ which is already contained in $L^1(G)$.

- If $(2 - x_0)y(2 - x_0)^{-1}$ would belong to $\mathbb{Q}G$, also the element $y(1 - x_1/2)(1 + x_0/2 + x_0^2/4 + \cdots)$ in $L^1(G)$ would belong to $\mathbb{Q}G$, what is obviously not true. Hence $(2 - x_0)y(2 - x_0)^{-1}$ in $\mathcal{D}(G)$ is not contained in $\mathbb{Q}G$.

- So the Dieudonné determinant of the matrix $A = \begin{pmatrix} 2 - x_0 & 1 \\ y & 0 \end{pmatrix}$ is represented by the element $(2 - x_0)y(2 - x_0)^{-1}$ which is not contained in $\mathbb{Q}G$ although all entries of A belong to $\mathbb{Z}G$.

Example ($G = \mathbb{Z}$)

- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathbb{Z}[\mathbb{Z}]) \cong (\mathbb{Q}[\mathbb{Z}]^{(0)})^\times.$$

- The following assertions are equivalent for a finite free $\mathbb{Z}[\mathbb{Z}]$-chain complex C_*:
  - C_* is weakly acyclic;
  - $\mathbb{Q}[\mathbb{Z}]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}]} C_*$ is acyclic;
  - $\mathbb{Q}[\mathbb{Z}]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}]} H_n(C_*)$ is trivial for every $n \in \mathbb{Z}$.
- Hence $\rho_u^{(2)}(C_*) \in (\mathbb{Q}[\mathbb{Z}]^{(0)})^\times$ is the alternating product of the Alexander polynomials of the $\mathbb{Z}[\mathbb{Z}]$-modules $H_n(C_*)$.

- Of course one can define the universal L^2-torsion for any G-covering $\overline{X}$ of a finite CW-complex X, provided that $b_n^{(2)}(\overline{X};\mathcal{N}(G)) = 0$ for every $n \in \mathbb{Z}$.
- It turns out that in the case $G = \mathbb{Z}$ the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory. So the Alexander polynomial can be deduced from the universal L^2-torsion applied to the infinite cyclic covering and all the formulas for the universal L^2-torsion carry over to the Alexander polynomial.
- We emphasize that already in the elementary cases $G = \{1\}$ and $G = \mathbb{Z}$ the universal L^2-torsion gives already very interesting invariants, namely the Milnor torsion and the Alexander polynomial.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$.
- Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by
$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$
- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion
$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Define the ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

by sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

- Its value at $t = 1$ is independent of ϕ and just the L^2-torsion $\rho^{(2)}(\widetilde{M})$. Recall that for an (irreducible) 3-manifold M (with empty or incompressible toroidal boundary and infinite fundamental group) this is up to the factor $-1/6\pi$ the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition.

- It is not at all obvious that this definition makes sense.
- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.
- On the analytic side this corresponds for a closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_{\pi} \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

Theorem (Lück [13])

Suppose that $\widetilde{X}$ is L^2-acyclic.

1. *The L^2torsion function $\rho^{(2)} := \rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$ is well-defined;*
2. *The limits $\limsup_{t\to\infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\liminf_{t\to 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

$$\deg(X;\phi) \in \mathbb{R}$$

to be their difference.

- The proof of the result above is based on approximation techniques and a profound understanding of Mahler measures.
- Actually, if we consider the function

$$f\colon (0,\infty) \to (0,\infty), \quad t \mapsto \exp \circ \rho^{(2)}(\widetilde{M};\phi)$$

  for an admissible 3-manifold M, then f is multiplicatively convex, i.e., $f(t_o^{\lambda} \cdot t_1^{1-\lambda}) \le f(t_0)^{\lambda} \cdot f(t_1)^{1-\lambda}$ for $t_0, t_1 \in (0,\infty)$ and $\lambda \in (0,1)$.
- In particular f and $\rho^{(2)}(\widetilde{M};\phi)$ are monotone increasing and continuous for a 3-manifold M.

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_-(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle with connected closed surface $F \ncong S^2$ and $\phi = \pi_1(p)$, then

$$x_M(\phi) = -\chi(F).$$

Theorem (Friedl-Lück [3], Liu [11])

Let M be an admissible 3-manifold. Then for every $\phi \in H^1(M;\mathbb{Z})$ we get the equality

$$\deg(M;\phi) = x_M(\phi).$$

- One can define by twisted Fuglede-Kadison determinants a group homomorphism

$$\mathrm{Wh}^w(\pi) \to \mathrm{map}((0,\infty),\mathbb{R})$$

  such that the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ is mapped to the L^2-torsion function.

- Hence there are sum formulas, fibration formulas and Poincaré duality for the L^2-torsion function, and the Thurston norm is determined by the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$.

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

- The Newton polytope
$$N(p) \subseteq \mathbb{R}^n$$
of a polynomial
$$p(t_1, t_2, \ldots, t_n) = \sum_{i_1,\ldots,i_n} a_{i_1,i_2,\ldots,i_n} \cdot t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$
in n variables $t_1, t_2, \ldots, t_n$ is defined to be the convex hull of the elements $\{(i_1, i_2, \ldots i_n) \in \mathbb{Z}^n \mid a_{i_1,i_2,\ldots,i_n} \neq 0\}$;
- One has
$$N(p \cdot q) = N(p) + N(q).$$

Definition (Polytope group)

- Let $\mathcal{P}_{\mathbb{Z}}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.
- Denote by $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ the quotient of $\mathcal{P}_{\mathbb{Z}}(A)$ by the canonical homomorphism $A \to \mathcal{P}_{\mathbb{Z}}(A)$ sending a to the class of the polytope $\{a\}$.

- In $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$ we consider polytopes up to translation with an element in A.
- Given a homomorphism of finitely generated abelian groups $f\colon A \to A'$, we obtain a homomorphisms of abelian groups

$$\mathcal{P}_{\mathbb{Z}}(f)\colon \mathcal{P}_{\mathbb{Z}}(A) \to \mathcal{P}_{\mathbb{Z}}(A'), \quad [P] \mapsto [\mathrm{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f(P)];$$

  and analogously for $\mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(A)$.

Example ($A = \mathbb{Z}$)

- An integral polytope in $\mathbb{Z}_{\mathbb{R}}$ is just an interval $[m, n]$ for $m, n \in \mathbb{Z}$ satisfying $m \leq n$.
- The Minkowski sum becomes $[m_1, n_1] + [m_2, n_2] = [m_1 + m_2, n_1 + n_2]$.
- One obtains isomorphisms of abelian groups

$$\begin{array}{rcll} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}^2 & [[m,n]] \mapsto (n-m, m). \\ \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z}, & [[m,n]] \mapsto n-m. \end{array}$$

Theorem (Funke [7])

$\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n)$ is a free abelian group possessing an explicite basis.

- A basis for $\mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^2)$ is given by the set comprising the 1-dimensional polytopes which are not proper multiples of another integral polytope, and rectangular triangles.
- We obtain an injection, useful for detection,

$$\mathcal{P}_{\mathbb{Z}}(A) \to \prod_{\phi \in \hom_{\mathbb{Z}}(A,\mathbb{Z})} \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}), \quad x \mapsto \big(\phi(x)\big)_{\phi}.$$

- We obtain a well-defined homomorphism of abelian groups

$$\big(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\big)^{\times} \to \mathcal{P}_{\mathbb{Z}}(\mathbb{Z}^n), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

  We want to generalize it to the so called polytope homomorphism.

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

  Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

  where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f, u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f.$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in \mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(\left(S^{-1}\left(\mathcal{D}(K) * H_1(G)_f\right)\right)^{\times} \right)_{\text{abel}} \to \mathcal{P}_{\mathbb{Z}}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- The composite

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \big(\mathcal{D}(G)^\times\big)_{\text{abel}}$$
$$\xrightarrow{\cong} \Big(\big(S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big)\big)^\times\Big)_{\text{abel}} \xrightarrow{P'} \mathcal{P}_{\mathbb{Z}}(H_1(G)_f)$$

factories to the polytope homomorphism

$$P\colon \operatorname{Wh}^w(G) \to \mathcal{P}_{\mathbb{Z},\text{Wh}}(H_1(G)_f).$$

Seminorms and compact convex subsets

- Let V be a finite-dimensional real vector space. We write $V^* = \hom(V, \mathbb{R})$.
- Given any subset X of V, we define its dual to be

$$X^* := \{\phi \in V^* \mid \phi(v) \leq 1 \text{ for all } v \in X\}.$$

- Given a compact convex subset $X \subseteq V$ we define a seminorm

$$\| \ \|_X \colon V^* \to [0, \infty), \quad \phi \mapsto \|\phi\|_X := \frac{1}{2} \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in X\}.$$

- We define $(-X)$ to be the compact convex subset $\{-x \mid x \in X\}$.
- The Minkowski sum $X + (-X)$ is again a compact convex subset and we get $\| \ \|_X = \| \ \|_{-X}$ and $2 \cdot \| \ \|_X = \| \ \|_{X+(-X)}$.
- Given a seminorm s on V, we assign to it its unit ball

$$B_s := \{v \in V \mid s(v) \leq 1\}$$

and denote by B_s^* the associated dual.

A straightforward argument shows that we have the equality

$$B_s^* = \{\phi \in V^* \mid \phi(v) \leq s(v) \text{ for all } v \in V\}.$$

- In the sequel we will identify $V = V^{**}$ by the canonical isomorphism.

Lemma

If X is a closed convex subset of V containing 0, then $X = X^{**}$.

Proof.

- It is straightforward to see that $X \subset X^{**}$. We now show the reverse inclusion.
- Consider $y \in V$ with $y \notin X$.
- By the Separating Hyperplane Theorem we can find $\psi \in V^*$ and $r \in \mathbb{R}$ such that $\psi(x) < r$ holds for all $x \in X$ and $\psi(y) > r$.
- Since 0 is contained in X and since $\psi(0) = 0$ we deduce that $r > 0$. Define $\phi := r^{-1} \cdot \psi \in V^*$. Then $\phi(x) \leq 1$ for $x \in X$ and $\phi(y) > 1$. This implies $y \notin X^{**}$.

□

Example

- Consider $X = [-1,1] \times \mathbb{R} \subseteq \mathbb{R}^2$,
- If we identify $(\mathbb{R}^2)^* = \mathbb{R}^2$ using the dual of the standard basis, we get

$$\begin{aligned} X^* &= \{(a,b) \in \mathbb{R}^2 \mid ax + by \le 1 \text{ for } (x,y) \in X\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid ax+by \le 1 \text{ for } x \in [-1,1], y \in \mathbb{R}\} = [-1,1]\times\{0\}. \end{aligned}$$

- We compute

$$\begin{aligned} (X^*)^* &= \{(x,y) \in \mathbb{R}^2 \mid xa + yb \le 1 \text{ for } (a,b) \in X^*\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid xa \le 1 \text{ for } a \in [-1,1],\} = [-1,1] \times \mathbb{R} = X. \end{aligned}$$

Lemma

Let s be a seminorm on V and $X \subseteq V$ be a compact convex subset. Then

- *The convex set B_s is compact if and only if s is a norm;*
- *B_s^* is convex and compact;*
- *For any $v \in V$ we have*

$$s(v) = \frac{1}{\sup\{r \in [0,\infty) \mid rv \in B_s\}};$$

- *We have $B_s = (B_s^*)^*$;*
- *We have $\| \ \|_{B_s^*} = s$;*
- *We have $X + (-X) = (B_{\| \ \|_X})^*$.*

Example

- Consider the norm $\|(x,y)\| = \max\{|x|,|y|\}$ on $\mathbb{R}^2$.
- Its unit ball is $B = [-1,1] \times [-1,1] \subseteq \mathbb{R}^2$.
- The dual of B is the subset $B^* \subset (\mathbb{R}^2)^* = \mathbb{R}^2$

$$\begin{aligned} B^* &= \{(a,b) \in \mathbb{R}^2 \mid ax + by \leq 1 \text{ for } (x,y) \in B\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid ax + by \leq 1 \text{ for } x,y \in [-1,1]\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid |a| + |b| \leq 1\}. \end{aligned}$$

- We compute for $(B^*)^* \subseteq \mathbb{R}^2$

$$\begin{aligned} (B^*)^* &= \{(x,y) \in \mathbb{R}^2 \mid xa + yb \leq 1 \text{ for } (a,b) \in B\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid xa + yb \leq 1 \text{ for } a,b \in \mathbb{R}, |a| + |b| \leq 1\} \\ &= \{(x,y) \in \mathbb{R}^2 \mid x,y \in [-1,1]\} = B. \end{aligned}$$

Example

- We compute for $(x, y) \in \mathbb{R}^2$

$$\begin{aligned}
\|(x,y)\|_{B^*} &= \frac{1}{2}\sup\{xa + yb \mid a, b \in B^*\} \\
&= \frac{1}{2}\sup\{xa + yb \mid (a,b) \in \mathbb{R}^2 \mid |a| + |b| \leq 1\} \\
&= \max\{|x|, |y|\} \\
&= \|(x,y)\|.
\end{aligned}$$

- Let M be a compact oriented 3-manifold.
- In the sequel we will identify $\mathbb{R} \otimes_{\mathbb{Z}} H_1(M)_f = H_1(M;\mathbb{R})$ and $H^1(M;\mathbb{R}) = H_1(M;\mathbb{R})^*$ by the obvious isomorphisms.
- We refer to
$$B_{x_M} := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leq 1\}$$
as the Thurston norm ball.

Definition (Dual Thurston polytope)

- We refer to
$$T(M)^* := B^*_{x_M} \subset (H^1(M;\mathbb{R}))^* = H_1(M;\mathbb{R})$$
as the dual Thurston polytope.
- Explicitly we have
$$T(M)^* = \{v \in H_1(M;\mathbb{R}) \mid \phi(v) \leq x_M(\phi) \text{ for all } \phi \in H^1(M;\mathbb{R})\}.$$

- Thurston [17] has shown that $T(M)^*$ is an integral polytope.
- Moreover, Thurston [17] showed that we can find a (possibly empty) marking on the vertices of $T(M)^*$ so that a cohomology class $\phi \in H^1(M;\mathbb{R})$ fibers, i.e., is represented by a non-degenerate closed 1-form, if and only if it pairs maximally with a marked vertex, i.e., if and only if there exists a marked vertex v such that $\phi(v) > \phi(w)$ for all $w \in TM^*$ with $v \neq w$. If ϕ lies in $H^1(X;\mathbb{Z})$, then fibers means that ϕ is induced by a surface bundle $F \to M \to S^1$.
- The Thurston seminorm x_M obviously determines the dual Thurston polytope.
- The converse is also true, namely, we have

$$x_M(\phi) := \frac{1}{2} \cdot \sup\{\phi(x_0) - \phi(x_1) \mid x_0, x_1 \in T(M)^*\}.$$

Definition (L^2-polytope)

Let X be a connected finite CW-complex such that $b_n^{(2)}(\widetilde{X}) = 0$ holds for all $N \geq 0$. Define its L^2-polytope $P^{(2)}(X)$ to be the element of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ under the polytope homomorphism

$$P\colon \operatorname{Wh}^w(\pi_1(M)) \to \mathcal{P}_{\mathbb{Z},\operatorname{Wh}}(H_1(\pi_1(M))_f).$$

- The name L^2-polytope $\rho_u^{(2)}(\widetilde{M})$ should not be misunderstood in the sense that it is a polytope. It is just an equivalence class of formal differences $[P] - [Q]$ of integral polytopes in $H_1(M;\mathbb{R}) = \mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f$, where we call $[P] - [Q]$ and $[P'] - [Q']$ equivalent if the two polytopes given by the Minkowski sums $P + Q'$ and $P' + Q$ in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f$ become equal after a translation with an element in $H_1(\pi_1(M))_f$.

- If one has $[P] = [P'] + [Q']$ in $\mathcal{P}_{\mathbb{Z},\text{Wh}}(H_1(\pi_1(M))_f)$ then one can read off $[P]$ from $[P'] - [Q']$ up to translation with elements in $H_1(\pi_1(M))_f$.
- Namely, if P_0 and P_1 are polytopes and v_0 and v_1 are elements in $H_1(\pi_1(M))_f$ with $P_0 + v_0 + Q' = P' = P_1 + v_1 + Q' = P'$, then we conclude $P_0 + (v_0 - v_1) = P_1$ since the Minkowski sum is cancellative.

Theorem (Friedl-Lück [4])

Let M be an admissible 3-manifold.

Then the L^2-polytope $P^{(2)}(X)$ is represented by the dual Thurston polytope $T(M)^*$.

- Since the dual Thurston polytope $T(M)^*$ satisfies $T(M)^* = -T(M)^*$, the L^2-polytope $P^{(2)}(X)$ determines $T(M)^*$ uniquely.

- This follows from the consideration above and the fact for two polytopes P and Q satisfying $P = -P$, $Q = -Q$ and $P = Q + v$ for some $v \in H_1(M)_f$, we must have $P = Q$.

- One can read of the universal L^2-torsion from a balanced presentation of the fundamental group of an admissible 3-manifold using Fox derivatives.

- Hence, by the theorem above, the same is true for the Thurston norm and the dual Thurston polytop.

- The problem in calculating the Thurston polytope explicitly is to identify the image of the universal L^2-torsion under the polytope homomorphism. There the main difficulty is the computation of the Dieudonné determinant.

- This is very easy for $(1, 1)$-matrices. Therefore one obtains rather explicite formulas for the polytope homomorphism for one-relator groups with two relations as described in Friedl-Tillmann [6].

L^2-Euler characteristic

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n \geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its **L^2-Euler characteristic**

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.
- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.
- It turns out that the ϕ-L^2-Euler characteristic is always an integer.

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f};\phi) = \chi(X).$$

Theorem (Friedl-Lück [4])

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\widetilde{M};\phi) = x_M(\phi).$$

- Suppose that G is torsionfree and satisfies the Atiyah Conjecture. Consider $\phi\colon G \to \mathbb{Z}$.

  Then there is a homomorphism

  $$\chi^{(2)}_{\phi}\colon \mathrm{Wh}^w(G) \to \mathbb{Z}$$

  which sends the universal L^2-torsion $\rho_u^{(2)}(\widetilde{X})$ to $\chi^{(2)}(\widetilde{X};\phi)$.

- Let $K \subset S^3$ be a knot.
- Let M_K be the complement of the interior of a tubular neighborhood of K in S^3. The knot group is $\pi_1(M_K) \cong \pi_1(M \setminus K)$.
- We refer to the minimal genus of a Seifert surface of K as the genus $g(K)$ of K.
- We have $H^1(M_K;\mathbb{Z}) \cong \mathbb{Z}$ and one easily checks that for any generator ϕ of $H^1(M_K;\mathbb{Z})$ we have

$$x_{M_K}(\phi) \;=\; \max\{2g(K) - 1, 0\}.$$

- One of the key motivations for developing the theory of L^2-Euler characteristics is the following question by Simon.

Question (Simon)

Let K and K' be two knots. If there is an epimorphism from the knot group of K to the knot group of K', does this imply that the genus of K is greater than or equal to the genus of K'?

- We propose the following conjecture.

Conjecture (Inequality of the Thurston norm)

Let $f\colon M \to N$ be a map between admissible 3-manifolds. Suppose that it induces an epimorphism $\pi_1(M) \to \pi_1(N)$ and an isomorphism $H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$.

Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

- A proof of the conjecture above would give an affirmative answer to Simon's question.
- The condition on the induced map on rational homology cannot be dropped.
- A group G is called locally indicable if any finitely generated non-trivial subgroup of G admits an epimorphism onto $\mathbb{Z}$. For example Howie [9] showed that the fundamental group of any admissible 3-manifold with non-trivial boundary is locally indicable.
- The next theorem is based on the theorem above about the equality of the Thurston norm and the L^2-Euler characteristic.

Theorem (Inequality of the Thurston norm)

Let $f\colon M \to N$ be a map of admissible 3-manifolds which is surjective on $\pi_1(N)$ and induces an isomorphism $f_*\colon H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$. Suppose that $\pi_1(N)$ is residually locally indicable elementary amenable. Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

Conjecture

The fundamental group of any admissible 3-manifold M with $b_1(M) \geq 1$ is residually locally indicable elementary amenable.

- A proof of the conjecture above together with the theorem above implies the Conjecture about the Inequality of the Thurston norm and in particular an affirmative answer to Simon's Question.

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.
- It has nice properties, e.g., it depends only on the conjugacy class of f, satisfies a sum formula and a formula for exact sequences.
- If G is amenable, it vanishes.
- If G is the fundamental group of a compact surface F and f comes from an automorphism $a\colon F \to F$, then T_f is a 3-manifold and a lot of the material above applies.
- For instance, if a is irreducible, $\rho_u^{(2)}(f)$ detects whether a is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

  Let

$$P(f) \in \mathcal{P}_{\mathbb{Z}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

  be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}_{\mathbb{Z}}(\mathrm{pr})} \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest. Connections to the Bieri-Neumann-Strebel invariants are expected.

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey [8] and Cochran [2], provided that G occurs in the rational derived series of $\pi_1(M)$.
- At least the degree of these polynomials is a well-defined invariant of M and G.
- Friedl-Lück [5] can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

A. Bartels, F. T. Farrell, and W. Lück.
The Farrell-Jones Conjecture for cocompact lattices in virtually connected Lie groups.
J. Amer. Math. Soc., 27(2):339–388, 2014.

T. D. Cochran.
Noncommutative knot theory.
Algebr. Geom. Topol., 4:347–398, 2004.

S. Friedl and W. Lück.
The L^2-torsion function and the Thurston norm of 3-manifolds.
Preprint, arXiv:1510.00264 [math.GT], to appear in Commentarii Mathematici Helvetici, 2015.

S. Friedl and W. Lück.
L^2-Euler characteristics and the Thurston norm.
Preprint, arXiv:1609.07805 [math.GT], 2016.

S. Friedl and W. Lück.
Universal L^2-torsion, polytopes and applications to 3-manifolds.
Proc. Lond. Math. Soc. (3), 114(6):1114–1151, 2017.

S. Friedl and S. Tillmann.
Two-generator one-relator groups and marked polytopes.
Preprint, arXiv:1501.03489v1 [math.GR], 2015.

F. Funke.
The integral polytope group.
Preprint, arXiv:1605.01217 [math.MG], 2016.

S. L. Harvey.
Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm.
Topology, 44(5):895–945, 2005.

J. Howie.
On locally indicable groups.
Math. Z., 180(4):445–461, 1982.

P. A. Linnell.
Division rings and group von Neumann algebras.
Forum Math., 5(6):561–576, 1993.

Y. Liu.
Degree of L^2-Alexander torsion for 3-manifolds.
Invent. Math., 207(3):981–1030, 2017.

J. Lott and W. Lück.
L^2-topological invariants of 3-manifolds.
Invent. Math., 120(1):15–60, 1995.

W. Lück.
Twisting L^2-invariants with finite-dimensional representations.
Preprint, arXiv:1510.00057 [math.GT], to appear in Journal of Topology and Analysis, 2015.

W. Lück and P. Linnell.
Localization, Whitehead groups and the Atiyah conjecture.
Ann. K-Theory, 3(1):33–53, 2018.

W. Lück and T. Schick.
L^2-torsion of hyperbolic manifolds of finite volume.
GAFA, 9:518–567, 1999.

T. Schick.
L^2-determinant class and approximation of L^2-Betti numbers.
Trans. Amer. Math. Soc., 353(8):3247–3265 (electronic), 2001.

W. P. Thurston.
A norm for the homology of 3-manifolds.
Mem. Amer. Math. Soc., 59(339):i–vi and 99–130, 1986.